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Global Dynamics of Classical Solutions to a Model of Mixing Flow

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Notes

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I. Introduction

As one of the core questions in mathematical fluid dynamics, the large-time asymptotic behavior of solutions to Cauchy problem or initial-boundary value problems for modeling equations is of central interest to researchers. Not only is the question physically important, it is also mathematically challenging. Positive answer to this question will undoubtedly benefit mathematicians, physicists and engineers. As is well known, the Navier-Stokes equations (NSE) have been one of the most important modeling systems in mathematical fluid dynamics for more than one hundred years. The comprehension of quantitative and qualitative behavior of the NSE plays an important role in understanding core problems in fluid mechanics, such as the onset of turbulence.

In this paper, we consider the following system of equations:

$$
\text{(MF)}\quad \begin{cases} (\rho U)_t + \nabla\cdot(\rho U\otimes U) + \nabla P = \nabla\cdot\big(\mu\nabla U - \lambda\rho[(\nabla U)+(\nabla U)^{\mathrm{T}}] + \nabla(\lambda\rho U)\big) + \\ \qquad\qquad\qquad\qquad\qquad\qquad \nabla\big(\nabla\cdot(\lambda\rho U)\big) + \rho\vec{f}, \\ \rho_t + \nabla\cdot(\rho U) = \lambda\Delta\rho, \\ \nabla\cdot U = 0, \end{cases}
$$

which describes the motion of an incompressible two-component mixture under the influence of external forces, with a diffusive mass exchange among the medium particles of various density accounted for [2]. Here, ρ is the density of the mixture, $U = (u, v)$ is the mean velocity, the constants $\mu > 0$ and $\lambda > 0$ model viscous dissipation and mass exchange, respectively, and $\vec{f}$ stands for external forces. For classical solutions, using the second and third equations, (MF) can be simplified to

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$$\begin{cases} \rho(U_t + U\cdot\nabla U) + \nabla P = \lambda\big(\nabla\rho\cdot\nabla U + U\cdot\nabla(\nabla\rho)\big) + \mu\Delta U + \rho\vec{f}, \\ \rho_t + U\cdot\nabla\rho = \lambda\Delta\rho, \\ \nabla\cdot U = 0. \end{cases} \tag{1.1}$$

System (1.1) generalizes the standard density-dependent incompressible Navier-Stokes equations for non-homogeneous fluid flows:

$$\begin{cases} \rho(U_t + U\cdot\nabla U) + \nabla P = \mu\Delta U + \rho\vec{f}, \\ \rho_t + U\cdot\nabla\rho = 0, \\ \nabla\cdot U = 0, \end{cases} \tag{NS}$$

which are important in applied fields of fluid dynamics such as oceanology and hydrology, and have been well-studied. We refer the reader to [2, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15] and references therein for details. It should be pointed out that a characteristic mathematical feature of (1.1) is its non-diagonality in its main part, which significantly distinguishes itself from (NS).

In real world, flows often move in bounded domains with constraints from boundaries, where initial-boundary value problems appear. Solutions to initial-boundary value problems usually exhibit different behaviors and much richer phenomena comparing with the Cauchy problem. In this paper, we consider (1.1) on a bounded domain in $\mathbb{R}^2$, and the system is supplemented by the following initial and boundary conditions:

$$\begin{cases} (U,\rho)(\mathbf{x},0) = (U_0,\rho_0)(\mathbf{x}), \quad m \le \rho_0(\mathbf{x}) \le M; \\ U|_{\partial\Omega} = 0, \quad \nabla\rho\cdot\mathbf{n}|_{\partial\Omega} = 0, \end{cases} \tag{1.2}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded domain with smooth boundary $\partial\Omega$, $\mathbf{n}$ is the unit outward normal to $\partial\Omega$ and m, M are positive constants.

It is well-known that classical solutions to (1.1)–(1.2) exist globally (locally resp.) in time in 2D (3D resp.) (c.f. [2]). However, to the best of the author's knowledge, the large-time asymptotic behavior of the solutions is not well-understood in the literature. In particular, the dynamics of the higher order modes of the solutions is not known. The purpose of this paper is to show that, under certain conditions on the external forcing term $\vec{f}$, the constant equilibrium state $(\bar{\rho}, \mathbf{0})$ is a global attractor of (1.1)–(1.2), for large initial data. Additionally, it is shown that the total Sobolev norm of the perturbation $(\rho-\bar{\rho}, U-\mathbf{0})$ up to the highest oder of derivatives converges exponentially in time due to the boundary effects. Here, $\bar{\rho}$ is the spatial average of ρ over Ω, which is a constant due to the conservation of total mass. The proof requires intensive applications of classical inequalities (Sobolev, Gagliardo-Nirenberg type) and tremendous amount of accurate energy estimates.

Throughout this paper, $\|\cdot\|_{L^p}$, $\|\cdot\|_{L^\infty}$ and $\|\cdot\|_{W^{s,p}}$ denote the norms of the usual Lebesgue measurable function spaces L^p $(1 \le p < \infty)$, L^∞ and the usual Sobolev space $W^{s,p}$, respectively. For $p = 2$, we denote the norm $\|\cdot\|_{L^2}$ by $\|\cdot\|$ and $\|\cdot\|_{W^{s,2}}$ by $\|\cdot\|_{H^s}$. For simplicity, we will use the following notation: $\|(f_1, f_2, ..., f_m)\|_X \equiv \sum_{i=1}^m \|f_i\|_X$. The

Ref.

[5] R. Danchin, Density-dependent incompressible fluids in critical spaces. *Proc. Royal Soc. Edinburgh* **133** (2003): 1311–1334.
[6] R. Danchin, Navier-Stokes equations with variable density. *Hyperbolic Problems and Related Topics, International Press, Graduate Series in Analysis* (2003): 121–135.

function spaces under consideration are $C([0,T];H^3(\Omega))$ and $L^2([0,T];H^4(\Omega))$, equipped with norms $\sup_{0\le t\le T}\|\Psi(\cdot,t)\|_{H^3}$ and $\left(\int_0^T\|\Psi(\cdot,t)\|_{H^4}^2dt\right)^{1/2}$, respectively. Unless specified, c_i will denote generic constants which are independent of ρ, U and t, but may depend on $\Omega, \lambda, \mu, M, m, \rho_0$ and U_0.

Our main results are summarized in the following theorem.

Notes

Theorem 1.1. *Let $\Omega\subset\mathbb{R}^2$ be a bounded domain with smooth boundary and suppose that the constant $\mu_1 = 2\mu - \lambda(M-m) > 0$. Suppose that the external force $\vec{f}$ is a potential flow, i.e., $\vec{f} = \nabla\phi$ for some scalar function $\phi : \Omega\times[0,\infty)\to\mathbb{R}$. Furthermore, suppose that there exists a constant $F_1 > 0$ independent of $t\ge 0$ such that $\|\vec{f}\|^2_{C([0,t];H^1(\Omega))} + \|\vec{f}\|^2_{L^2([0,t];H^2(\Omega))} + \|\vec{f}_t\|^2_{C([0,t];L^2(\Omega))} \le F_1$ for any $t\ge 0$. If the initial data $(\rho_0(\mathbf{x}), U_0(\mathbf{x}))\in H^3(\Omega)$ are compatible with the boundary conditions, then there exists a unique solution (ρ,U) to (1.1)–(1.2) globally in time such that $(\rho,U)(\mathbf{x},t)\in C([0,T);H^3(\Omega))\cap L^2([0,T);H^4(\Omega))$ for any $T\ge 0$. Moreover, there exist positive constants α,β and γ independent of t such that the solution satisfies*

$$\|(\rho-\bar{\rho},U)(\cdot,t)\|^2_{H^3}\le\alpha e^{-\beta t},\quad\text{and}\quad\int_0^t\|(\rho-\bar{\rho},U)(\cdot,\tau)\|^2_{H^4}d\tau\le\gamma,\quad\forall\ t\ge 0;$$
$$m\le\rho(\mathbf{x},t)\le M,\quad\forall\ t\ge 0,\ \mathbf{x}\in\Omega,$$

where m and M are given in (1.2).

Remark 1.1. *The external forcing term $\vec{f}$ includes important applications such as $\vec{f} = \mathbf{e}_2 = (0,1)^{\mathrm{T}}$, which stands for the effect of gravitational force. Physically speaking, the results indicate that, when the viscous dissipation dominates the mass exchange rate, as time goes on, the velocity of the flow will slow down and the mixture tends to be homogeneous.*

Remark 1.2. *The condition on the diffusion coefficients and the upper-lower bounds of the density can be roughly understood by looking at the stress tensor in the momentum equation in (MF), where competition between viscous dissipation and mass exchange occurs.*

Remark 1.3. *One can generalize the results by manipulating on various boundary conditions for ρ and U. For example, one can work on the Dirichlet boundary condition $\rho|_{\partial\Omega} = \tilde{\rho}$, for some constant $m\le\tilde{\rho}\le M$. In this case, the lower and upper bounds of ρ are direct consequences of maximum principle for parabolic equations, and the equilibrium state of ρ is $\tilde{\rho}$. On the other hand, the results may also be generalized to the Navier type slip boundary condition $U\cdot\mathbf{n}|_{\partial\Omega} = 0$, $\omega|_{\partial\Omega} = 0$, where ω is the 2D vorticity. Since the underlying analysis is in the similar fashion, we shall not go through the details in this paper.*

The main difficulties of the proof of Theorem 1.1 come from the estimation of the higher order derivatives of the solution, due to the coupling between the velocity and density equations by convection, diffusion, external force and boundary effects. With the density function and the additional nonlinear terms $\nabla\rho\cdot\nabla U$ and $U\cdot\nabla(\nabla\rho)$ standing in the velocity equation, the decay of the higher order derivatives of U is an substantial barrier to overcome. Great efforts have been made to simplify the proof. Current proof involves intensive applications of fundamental inequalities, together with exhaustive combinations of energy inequalities. The results on Stokes equation by Temam [17], see lemma 2.1, are important in our energy framework. Roughly speaking, because of the lack of the spatial derivatives of the solution at the boundary, our energy framework proceeds as follows: We first apply the standard energy estimate on the solution and the temporal derivatives of the solution. We then apply Temam's results on Stokes equation to recover the spatial derivatives. Such a process will be repeated up to the third order, and then the carefully coupled estimates will be composed into a desired one leading to global regularity and exponential decay of the solution. The condition $\vec{f} = \nabla\phi$ is crucial in our analysis due to the fact that, by combining $\bar{\rho}\nabla\phi$ with ∇P, the density perturbation on the right hand side of the velocity equation will be dominated by the diffusion in the density equation, by virtue of Poincaré inequality. This enables us to combine various energy estimates which eventually lead to the exponential decay of the solution. The result suggests that the diffusions are strong enough to compensate the effects of external force and nonlinear convection in order to prevent the development of singularity of the system and to force the solution to converge to the equilibrium state.

The rest of this paper is organized as follows. In Section 2, we give some basic facts that will be used in this paper. We then prove Theorem 1.1 in Section 3.

II. Preliminaries

In this section, we will list several facts which will be used in the proof of Theorem 1.1. First we recall some useful results from [17].

Lemma 2.1. *Let Ω be any open bounded domain in $\mathbb{R}^2$ with smooth boundary $\partial\Omega$. Consider the Stokes problem*

$$\begin{cases} -\mu\Delta U + \nabla P = F \quad \text{in } \Omega, \\ \nabla\cdot U = 0 \quad \text{in } \Omega, \\ U = 0 \quad \text{on } \partial\Omega. \end{cases}$$

If $F \in W^{m,p}$, then $U \in W^{m+2,p}$, $P \in W^{m+1,p}$ and there exists a constant $c_1 = c_1(\mu, m, p, \Omega)$ such that

$$\|U\|^2_{W^{m+2,p}} + \|P\|^2_{W^{m+1,p}} \le c_1\|F\|^2_{W^{m,p}}$$

for any $p \in (1,\infty)$ and the integer $m \ge -1$.

The next lemma will be used in the estimation of higher order spatial derivatives of ρ (c.f. [3]).

Lemma 2.2. *Let $\Omega \subset \mathbb{R}^2$ be any open bounded domain with smooth boundary $\partial\Omega$, and let $G \in W^{s,p}(\Omega)$ be a vector-valued function satisfying $\nabla\times G = 0$ and $G\cdot\mathbf{n}|_{\partial\Omega} = 0$, where $\mathbf{n}$ is the unit outward normal to $\partial\Omega$. Then there exists a constant $c_2 = c_2(s,p,\Omega)$ such that*

$$\|G\|^2_{W^{s,p}} \le c_2(\|\nabla\cdot G\|^2_{W^{s-1,p}} + \|G\|^2_{L^p})$$

for any $s \ge 1$ and $p \in (1,\infty)$.

As a consequence of Poincaré inequality and Lemma 2.2 we have

Lemma 2.3. *Let $\Omega \subset \mathbb{R}^2$ be any open bounded domain with smooth boundary $\partial\Omega$. For any function $H^s(\Omega) \ni f : \Omega \to \mathbb{R}$ satisfying $\nabla f\cdot\mathbf{n}|_{\partial\Omega} = 0$, let $\bar{f} = \frac{1}{|\Omega|}\int_\Omega f d\mathbf{x}$, where the*

Global Journal of Science Frontier Research (F) Volume XII Issue II Version I 26 February 2012

Ref.

[17] R. Temam, *Navier-Stokes Equations.* North Holland, 1977.

integer $s \geq 2$. *Then there exists a constant* $c_3 = c_3(\Omega, s)$ *such that*

$$\|f - \bar{f}\|_{H^s}^2 \leq c_3 \|\Delta f\|_{H^{s-2}}^2.$$

We also need the following Sobolev and Ladyzhenskaya type inequalities which are well-known and standard (c.f. [1, 4, 16]).

Lemma 2.4. *Let* $\Omega \subset \mathbb{R}^2$ *be any open bounded domain with smooth boundary* $\partial\Omega$. *Then the following embeddings and inequalities hold:*

(i) $\|f\|_{L^p}^2 \leq c_4 \|f\|_{H^1}^2, \quad \forall\ 1 < p < \infty;$

(ii) $\|f\|_{L^\infty}^2 \leq c_5 \|f\|_{W^{1,p}}^2, \quad \forall\ 2 < p < \infty;$

(iii) $\|f\|_{L^4}^2 \leq c_6 \|f\| \|\nabla f\|, \quad \forall\ f \in H_0^1(\Omega);$

(iv) $\|f\|_{L^4}^2 \leq c_7 \big(\|f\| \|\nabla f\| + \|f\|^2\big), \ \forall\ f \in H^1(\Omega),$

for some constants $c_i = c_i(p, \Omega),\ i = 4, ..., 7.$

III. LARGE-TIME BEHAVIOR

In this section we prove Theorem 1.1. Since the global existence has been established in [2], we only show the large-time behavior of the solution. The proof is based on several steps of careful energy estimates which are stated as a sequence of lemmas. First of all, the L^∞ estimate of ρ is a direct consequence of the maximum principle:

Lemma 3.1. *Under the assumptions of Theorem 1.1, it holds that*

$$m \leq \rho(\mathbf{x}, t) \leq M, \quad \forall\ t \geq 0,\ \mathbf{x} \in \Omega.$$

a) Reformulation

In order to perform the asymptotic analysis, we first reformulate the original problem (1.1)–(1.2) to get a new one for the perturbation $(\rho - \bar{\rho}, U)$. Letting $\theta = \rho - \bar{\rho}$ and $Q = P - \bar{\rho}\phi$ we have

$$\begin{cases} \rho(U_t + U \cdot \nabla U) + \nabla Q = \lambda\big(\nabla\theta \cdot \nabla U + U \cdot \nabla(\nabla\theta)\big) + \mu\Delta U + \vec{f}\theta, \\ \theta_t + U \cdot \nabla\theta = \lambda\Delta\theta, \\ \nabla \cdot U = 0. \end{cases} \tag{3.1}$$

The initial and boundary conditions turn out to be

$$\begin{cases} (U, \theta)(\mathbf{x}, 0) = (U_0, \theta_0)(\mathbf{x}) \equiv (U_0, \rho_0 - \bar{\rho})(\mathbf{x}); \\ U|_{\partial\Omega} = 0, \quad \nabla\theta \cdot \mathbf{n}|_{\partial\Omega} = 0. \end{cases} \tag{3.2}$$

b) Decay of $\|(U, \theta)\|$

Lemma 3.2. *Under the assumptions of Theorem 1.1, there exist positive constants* α_1, β_1 *and* γ_1 *independent of* t *such that for any* $t \geq 0$ *it holds that*

$$\|(U, \theta)(\cdot, t)\|^2 \leq \alpha_1 e^{-\beta_1 t}, \quad \text{and} \quad \int_0^t \|(U, \theta)(\cdot, \tau)\|_{H^1}^2 d\tau \leq \gamma_1.$$

Ref.

[2] S.N. Antontsev, A.V. Kazhikhov and V.N. Monakhov, *Boundary value problems in mechanics of nonhomogeneous fluids.* North-Holland, 1990.

Proof. The lemma is proved through careful exploration of the structure of the system. First of all, by taking L^2 inner product of $(3.1)_1$ with U we have

$$\int_\Omega \rho\Big(\frac{|U|^2}{2}\Big)_t d\mathbf{x} + \int_\Omega \rho U\cdot\nabla\Big(\frac{|U|^2}{2}\Big)d\mathbf{x} + \mu\int_\Omega |\nabla U|^2 d\mathbf{x}$$
$$=\lambda\int_\Omega \nabla\theta\cdot\nabla\Big(\frac{|U|^2}{2}\Big)d\mathbf{x} + \lambda\int_\Omega \big(U\cdot\nabla(\nabla\theta)\big)\cdot U d\mathbf{x} + \int_\Omega \theta\vec{f}\cdot U d\mathbf{x}.$$

After integration by parts and using the incompressibility condition we have

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \rho|U|^2 d\mathbf{x} - \frac{1}{2}\int_\Omega \theta_t|U|^2 d\mathbf{x} - \frac{1}{2}\int_\Omega \nabla\cdot(\theta U)|U|^2 d\mathbf{x} + \mu\int_\Omega |\nabla U|^2 d\mathbf{x}$$
$$= -\frac{\lambda}{2}\int_\Omega \Delta\theta|U|^2 d\mathbf{x} + \lambda\int_\Omega \big(U\cdot\nabla(\nabla\theta)\big)\cdot U d\mathbf{x} + \int_\Omega \theta\vec{f}\cdot U d\mathbf{x}.$$

Using $(3.1)_2$ we simplify the above equation as

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \rho|U|^2 d\mathbf{x} + \mu\int_\Omega |\nabla U|^2 d\mathbf{x} = \lambda\int_\Omega \big[U\cdot\nabla(\nabla\theta)\big]\cdot U d\mathbf{x} + \int_\Omega \theta\vec{f}\cdot U d\mathbf{x}. \quad (3.3)$$

For the first term on the RHS of (3.3), by direct calculations we have

$$\big[U\cdot\nabla(\nabla\theta)\big]\cdot U = \nabla\cdot\big[U(U\cdot\nabla\theta) - (\theta U\cdot\nabla U)\big] + \theta(u_x^2 + 2u_y v_x + v_y^2). \quad (3.4)$$

Therefore, integrating (3.4) over Ω using the boundary condition we get

$$\int_\Omega \big[U\cdot\nabla(\nabla\theta)\big]\cdot U d\mathbf{x} = \int_\Omega \theta(u_x^2 + 2u_y v_x + v_y^2)d\mathbf{x}. \quad (3.5)$$

Using (3.5) we update (3.3) as

$$\frac{1}{2}\frac{d}{dt}\|\sqrt{\rho}U\|^2 + \mu\|\nabla U\|^2 = \lambda\int_\Omega \theta(u_x^2 + 2u_y v_x + v_y^2)d\mathbf{x} + \int_\Omega \theta\vec{f}\cdot U d\mathbf{x}. \quad (3.6)$$

Since $\nabla\cdot U = 0$, we have

$$u_x^2 + 2u_y v_x + v_y^2 = \nabla\cdot(U\cdot\nabla U) - U\cdot\nabla(\nabla\cdot U) = \nabla\cdot(U\cdot\nabla U),$$

which implies that

$$\int_\Omega (u_x^2 + 2u_y v_x + v_y^2)d\mathbf{x} = 0.$$

Since $\bar{\rho}$ is a constant, it follows from (3.6) and the above identity that

$$\frac{1}{2}\frac{d}{dt}\|\sqrt{\rho}U\|^2 + \mu\|\nabla U\|^2 = \lambda\int_\Omega \Big(\rho - \frac{M+m}{2}\Big)(u_x^2 + 2u_y v_x + v_y^2)d\mathbf{x} + \int_\Omega \theta\vec{f}\cdot U d\mathbf{x}. \quad (3.7)$$

Using Lemma 3.1 we estimate the RHS of (3.7) as follows:

$$\Big|\lambda\int_\Omega \Big(\rho - \frac{M+m}{2}\Big)(u_x^2 + 2u_y v_x + v_y^2)d\mathbf{x} + \int_\Omega \theta\vec{f}\cdot U d\mathbf{x}\Big|$$
$$\le\lambda\frac{M-m}{2}\|\nabla U\|^2 + \int_\Omega |\theta\vec{f}\cdot U|d\mathbf{x}.$$

Notes

We remark that the coefficient of $\|\nabla U\|^2$ on the RHS of the above estimate is optimal. So we update (3.7) as

$$\frac{d}{dt}\|\sqrt{\rho}U\|^2 + \mu_1\|\nabla U\|^2 \le 2\int_\Omega |\theta\vec{f}\cdot U|d\mathbf{x}, \tag{3.8}$$

where $\mu_1 = 2\mu - \lambda(M-m) > 0$. Using Cauchy-Schwarz and Poincaré inequalities we estimate the RHS of (3.8) as:

Notes

$$\begin{aligned} 2\int_\Omega |\theta\vec{f}\cdot U|d\mathbf{x} &\le \frac{\mu_1}{2c_0}\|U\|^2 + \frac{2c_0}{\mu_1}\|\vec{f}\theta\|^2 \\ &\le \frac{\mu_1}{2}\|\nabla U\|^2 + \frac{2c_0}{\mu_1}\|\vec{f}\theta\|^2. \end{aligned} \tag{3.9}$$

Since $\|\vec{f}\|^2_{C([0,t];H^1(\Omega))} \le F_1$, by Lemma 2.4 (i) we have

$$\begin{aligned} \frac{c_0}{2\mu_1}\|\vec{f}\theta\|^2 &\le \frac{c_0}{2\mu_1}\|\vec{f}\|^2_{L^4}\|\theta\|^2_{L^4} \\ &\le \frac{c_0c_4^2}{2\mu_1}\|\vec{f}\|^2_{H^1}\|\theta\|^2_{H^1} \\ &\le \frac{c_0c_4^2F_1}{2\mu_1}(1+c_0)\|\nabla\theta\|^2. \end{aligned} \tag{3.10}$$

Let $c_8 = c_0c_4^2F_1(1+c_0)/(2\mu_1)$. Combining (3.8)–(3.10) we have

$$\frac{d}{dt}\|\sqrt{\rho}U\|^2 + \frac{\mu_1}{2}\|\nabla U\|^2 \le c_8\|\nabla\theta\|^2. \tag{3.11}$$

The RHS of (3.11) will be compensated by the diffusion in the temperature equation. Taking L^2 inner product of $(3.1)_2$ with θ we have

$$\frac{d}{dt}\|\theta\|^2 + 2\lambda\|\nabla\theta\|^2 = 0. \tag{3.12}$$

Then the operation (3.12) $\times\, c_8/\lambda +$ (3.11) yields

$$\frac{d}{dt}\left(\frac{c_8}{\lambda}\|\theta\|^2 + \|\sqrt{\rho}U\|^2\right) + c_8\|\nabla\theta\|^2 + \frac{\mu_1}{2}\|\nabla U\|^2 \le 0. \tag{3.13}$$

Since $\rho \le M$, we have

$$\|\sqrt{\rho}U\|^2 \le M\|U\|^2 \le c_0M\|\nabla U\|^2.$$

It follows from (3.13) that

$$\frac{d}{dt}\left(\frac{c_8}{\lambda}\|\theta\|^2 + \|\sqrt{\rho}U\|^2\right) + \beta_1\left(\frac{c_8}{\lambda}\|\theta\|^2 + \|\sqrt{\rho}U\|^2\right) \le 0, \tag{3.14}$$

where

$$\beta_1 = \min\left\{\frac{\lambda}{c_0}, \frac{\mu_1}{2c_0M}\right\}. \tag{3.15}$$

Solving the differential inequality (3.14) we have

$$\left(\frac{c_8}{\lambda}\|\theta\|^2 + \|\sqrt{\rho}U\|^2\right) \le \left(\frac{c_8}{\lambda}\|\theta_0\|^2 + \|\sqrt{\rho_0}U_0\|^2\right)e^{-\beta_1 t}. \tag{3.16}$$

Since $\rho \geq m$, we get from (3.16) that

$$\|(U,\theta)(\cdot,t)\|^2 \leq \alpha_1 e^{-\beta_1 t}, \quad \forall\, t \geq 0, \tag{3.17}$$

where

$$\alpha_1 = \left(\min\left\{\frac{c_8}{\lambda}, m\right\}\right)^{-1}\left(\frac{c_8}{\lambda}\|\theta_0\|^2 + \|\sqrt{\rho_0}U_0\|^2\right). \tag{3.18}$$

Next, upon integrating (3.13) in time and dropping the positive term from the LHS we have

$$\int_0^t c_8\|\nabla\theta(\cdot,\tau)\|^2 + \frac{\mu_1}{2}\|\nabla U(\cdot,\tau)\|^2 d\tau \leq \frac{c_8}{\lambda}\|\theta_0\|^2 + \|\sqrt{\rho_0}U_0\|^2, \quad \forall\, t \geq 0,$$

which, together with (3.17), yields

$$\int_0^t \|(U,\theta)(\cdot,\tau)\|_{H^1}^2 d\tau \leq \gamma_1, \quad \forall\, t \geq 0, \tag{3.19}$$

where

$$\gamma_1 = \frac{\alpha_1}{\beta_1} + \left(\frac{c_8}{\lambda}\|\theta_0\|^2 + \|\sqrt{\rho_0}U_0\|^2\right)\left(\min\{c_8, \mu_1/2\}\right)^{-1}. \tag{3.20}$$

This completes the proof.

Remark 3.1. *The idea of the above proof will be appplied to prove the exponential decay of higher order derivatives of the solution. From (3.15) we see clearly that, the decay rate* β_1 *tends to zero, as either* λ *or* $\mu_1 = 2\mu - \lambda(M-m)$ *tends to zero. Furthermore, by (3.18) we have* $\alpha_1 \to \infty$, *as* $\lambda \to 0$ *or* $\mu_1 \to 0$. *Therefore, as the value of* λ *either decreases or approaches the threshold value* $\frac{2\mu}{M-m}$, *the decay of the solution will slow down. By direct calculation we know that the decay rate reaches its maximum when* $\lambda = \frac{2\mu}{3M-m}$.

Remark 3.2. *In what follows, since tremendous amount of combinations of energy estimates will be involved when we deal with the decay of higher order derivatives of the solution, the expressions of the constants appearing in the proofs will become lengthy and hard to read. Therefore, to simplify the presentation, we shall not specify* $c_i, \alpha_i, \beta_i, \gamma_i$ *in terms of the other time-independent constants.*

c) Decay of $\|\theta\|_{H^1}$

Lemma 3.3. *Under the assumptions of Theorem 1.1, there exist positive constants* α_2, β_2 *and* γ_2 *independent of* t *such that for any* $t \geq 0$ *it holds that*

$$\|\nabla\theta(\cdot,t)\|^2 \leq \alpha_2 e^{-\beta_2 t}, \quad \text{and} \quad \int_0^t \|\Delta\theta(\cdot,\tau)\|^2 + \|\theta_t(\cdot,\tau)\|^2 d\tau \leq \gamma_2.$$

Proof. Taking L^2 inner product of $(3.1)_2$ with $\Delta\theta$ we have

$$\frac{1}{2}\frac{d}{dt}\|\nabla\theta\|^2 + \lambda\|\Delta\theta\|^2 = \int_\Omega (U\cdot\nabla\theta)\Delta\theta\, d\mathbf{x}. \tag{3.21}$$

Using Cauchy-Schwarz and Hölder inequalities we estimate the RHS of (3.21) as

$$\begin{aligned}\left|\int_\Omega (U\cdot\nabla\theta)\Delta\theta d\mathbf{x}\right| &\leq \frac{1}{\lambda}\|U\cdot\nabla\theta\|^2 + \frac{\lambda}{4}\|\Delta\theta\|^2 \\ &\leq \frac{1}{\lambda}\|U\|_{L^4}^2\|\nabla\theta\|_{L^4}^2 + \frac{\lambda}{4}\|\Delta\theta\|^2.\end{aligned}$$

Notes

So we update (3.21) as

$$\frac{1}{2}\frac{d}{dt}\|\nabla\theta\|^2+\frac{3}{4}\lambda\|\Delta\theta\|^2\le\frac{1}{\lambda}\|U\|_{L^4}^2\|\nabla\theta\|_{L^4}^2. \tag{3.22}$$

Applying Lemma 2.4 (iii) to the RHS of (3.22) we have

$$\frac{1}{\lambda}\|U\|_{L^4}^2\|\nabla\theta\|_{L^4}^2\le c_9\|U\|\|\nabla U\|\|\nabla\theta\|\|D^2\theta\|+c_9\|U\|\|\nabla U\|\|\nabla\theta\|^2. \tag{3.23}$$

For the first term on the RHS of (3.23), using Lemma 2.3 for $\|D^2\theta\|^2$ and Lemma 3.2 for $\|U\|^2$ we have

$$\begin{aligned}c_9\|U\|\|\nabla U\|\|\nabla\theta\|\|D^2\theta\|&\le c_{10}\|\nabla U\|\|\nabla\theta\|\|\Delta\theta\|\\&\le c_{11}\|\nabla U\|^2\|\nabla\theta\|^2+\frac{\lambda}{4}\|\Delta\theta\|^2.\end{aligned} \tag{3.24}$$

Applying Poincaré inequality to the second term on the RHS of (3.23) we have

$$c_9\|U\|\|\nabla U\|\|\nabla\theta\|^2\le c_{12}\|\nabla U\|^2\|\nabla\theta\|^2. \tag{3.25}$$

Combining (3.23)–(3.25) we have

$$\frac{1}{\lambda}\|U\|_{L^4}^2\|\nabla\theta\|_{L^4}^2\le c_{13}\|\nabla U\|^2\|\nabla\theta\|^2+\frac{\lambda}{4}\|\Delta\theta\|^2. \tag{3.26}$$

Plugging (3.26) into (3.22) we have

$$\frac{1}{2}\frac{d}{dt}\|\nabla\theta\|^2+\frac{\lambda}{2}\|\Delta\theta\|^2\le c_{13}\|\nabla U\|^2\|\nabla\theta\|^2. \tag{3.27}$$

Gronwall inequality and Lemma 3.2 then yield (by dropping $\frac{\lambda}{2}\|\Delta\theta\|^2$)

$$\|\nabla\theta(\cdot,t)\|^2\le\exp\left\{2c_{13}\int_0^t\|\nabla U\|^2d\tau\right\}\|\nabla\theta_0\|^2\le e^{2c_{13}\gamma_1}\|\nabla\theta_0\|^2\equiv c_{14}. \tag{3.28}$$

Plugging (3.28) into (3.27) we have

$$\frac{1}{2}\frac{d}{dt}\|\nabla\theta\|^2+\frac{\lambda}{2}\|\Delta\theta\|^2\le c_{15}\|\nabla U\|^2. \tag{3.29}$$

To deal with the RHS of (3.29), we consider the estimate (3.13). The combination (3.13) $\times\ \frac{4c_{15}}{\mu_1}$ + (3.29) gives

$$\frac{d}{dt}\big(E_1(t)\big)+\frac{4c_8c_{15}}{\mu_1}\|\nabla\theta\|^2+c_{15}\|\nabla U\|^2+\frac{\lambda}{2}\|\Delta\theta\|^2\le 0, \tag{3.30}$$

where

$$E_1(t)=\frac{4c_{15}}{\mu_1}\left(\frac{c_8}{\lambda}\|\theta\|^2+\|\sqrt{\rho}U\|^2\right)+\frac{1}{2}\|\nabla\theta\|^2. \tag{3.31}$$

Using Poincaré inequality one easily checks that there exists a constant $\beta_2>0$ independent of t such that

$$\beta_2E_1(t)\le\left(\frac{4c_8c_{15}}{\mu_1}\|\nabla\theta\|^2+c_{15}\|\nabla U\|^2\right), \tag{3.32}$$

Using (3.32) we update (3.30) as

$$\frac{d}{dt}\big(E_1(t)\big)+\beta_2E_1(t)+\frac{\lambda}{2}\|\Delta\theta\|^2\le 0, \tag{3.33}$$

Notes

which implies that

$$E_1(t) \le e^{-\beta_2 t} E_1(0), \quad \text{and} \quad \frac{\lambda}{2}\int_0^t \|\Delta\theta(\cdot,\tau)\|^2 d\tau \le E_1(0), \quad \forall\, t \ge 0. \tag{3.34}$$

By (3.31) and (3.34) we see that

$$\|\nabla\theta(\cdot,t)\|^2 \le \alpha_2 e^{-\beta_2 t}, \quad \text{and} \quad \int_0^t \|\Delta\theta(\cdot,\tau)\|^2 d\tau \le 2E_1(0)/\lambda, \quad \forall\, t \ge 0, \tag{3.35}$$

where $\alpha_2 = 2E_1(0)$.

To estimate θ_t, we consider $(3.1)_2$. Using (3.26) and (3.35) we have

$$\begin{aligned} \|\theta_t\|^2 &\le 2\|U\cdot\nabla\theta\|^2 + 2\|\lambda\Delta\theta\|^2 \\ &\le 2\|U\|_{L^4}^2\|\nabla\theta\|_{L^4}^2 + 2\lambda^2\|\Delta\theta\|^2 \\ &\le c_{16}\big(\|\Delta\theta\|^2 + \|\nabla U\|^2\|\nabla\theta\|^2\big) + 2\lambda^2\|\Delta\theta\|^2 \\ &\le c_{17}\big(\|\Delta\theta\|^2 + \|\nabla U\|^2\big). \end{aligned} \tag{3.36}$$

Integrating (3.36) in time over $[0,t]$ and using Lemma 3.2 and (3.35) we have

$$\int_0^t \|\theta_t(\cdot,\tau)\|^2 d\tau \le c_{18}, \quad \forall\, t \ge 0. \tag{3.37}$$

We conclude the proof by combining (3.35) and (3.37).

d) Estimate of $\|U\|_{H}{}^2$

Now we turn to higher order estimates of the solution. The next lemma gives the control of $\|U\|_{H^2}$ by $\|\nabla U\|$, $\|U_t\|$ and estimates of θ. The proof involves intensive applications of Sobolev and Ladyzhenskaya type inequalities.

Lemma 3.4. *Under the assumptions of Theorem 1.1, for any positive numbers ε and δ, there exists a constant $d(\varepsilon,\delta)$ independent of t and dependent on ε and δ such that*

$$\|U\|_{H^2}^2 \le \delta\|\nabla\theta_t\|^2 + \varepsilon\|U\|_{H^2}^2 + d(\varepsilon,\delta)\big(\|\nabla U\|^2\|\theta\|_{H^2}^2 + \|\nabla U\|^4 + \|U_t\|^2 + \|\theta\|_{H^1}^2\big).$$

Proof. We rewrite the velocity equation $(3.1)_1$ as the 2D Stokes equation:

$$-\mu\Delta U + \nabla P = \vec{F},$$

where

$$\vec{F} = -\rho U_t - \rho U\cdot\nabla U + \lambda\nabla\theta\cdot\nabla U + \lambda U\cdot\nabla(\nabla\theta) + \vec{f}\theta \equiv \sum_{i=1}^{5} F_i.$$

Since $U|_{\partial\Omega} = 0$, it follows from Lemma 2.1 that

$$\|U\|_{H^2}^2 \le 16c_1\sum_{i=1}^{5}\|F_i\|^2. \tag{3.38}$$

Now we estimate the summand on the RHS of (3.38) as follows: Using Lemma 3.1 we have

$$\|F_1\|^2 = \|\rho U_t\|^2 \le M^2\|U_t\|^2. \tag{3.39}$$

Notes

Notes

Using Lemma 2.4 (iii), Lemma 3.1 and Lemma 3.3, we have, for any $\varepsilon > 0$:

$$\begin{aligned}\|F_2\|^2 &= \|\rho U\cdot\nabla U\|^2\\ &\le M^2\|U\|_{L^4}^2\|\nabla U\|_{L^4}^2\\ &\le c_{19}\|U\|\|\nabla U\|\big(\|\nabla U\|\|D^2U\|+\|\nabla U\|^2\big)\\ &\le c_{20}\|\nabla U\|^2\|U\|_{H^2}\\ &\le \frac{c_{21}}{\varepsilon}\|\nabla U\|^4+\frac{\varepsilon}{48c_1}\|U\|_{H^2}^2.\end{aligned} \tag{3.40}$$

For F_3, it follows from Lemma 3.3 that

$$\begin{aligned}\|F_3\|^2 &= \lambda^2\|\nabla\theta\cdot\nabla U\|^2\\ &\le c_{22}\big(\|\nabla\theta\|\|D^2\theta\|+\|\nabla\theta\|^2\big)\big(\|\nabla U\|\|D^2U\|+\|\nabla U\|^2\big)\\ &\le c_{23}\big(\|D^2\theta\|+\|\nabla\theta\|\big)\big(\|D^2U\|+\|\nabla U\|\big)\|\nabla U\|\\ &\le \frac{c_{24}}{\varepsilon}\|\theta\|_{H^2}^2\|\nabla U\|^2+\frac{\varepsilon}{48c_1}\|U\|_{H^2}^2.\end{aligned} \tag{3.41}$$

For the estimate of F_4, by Lemma 2.3 and Lemma 2.4 we have

$$\begin{aligned}\|F_4\|^2 &= \lambda^2\|U\cdot\nabla(\nabla\theta)\|^2\\ &\le c_{25}\|U\|\|\nabla U\|\|D^2\theta\|\big(\|D^3\theta\|+\|D^2\theta\|\big).\end{aligned} \tag{3.42}$$

To estimate $\|D^3\theta\|$, by Lemma 2.2 we have

$$\begin{aligned}\|D^3\theta\| &\le \sqrt{c_3}\|\Delta\theta\|_{H^1}\\ &\le c_{26}\big(\|\nabla\theta_t\|+\|\nabla(U\cdot\nabla\theta)\|+\|\Delta\theta\|\big)\\ &\le c_{27}\big(\|\nabla\theta_t\|+\|\nabla U\cdot(\nabla\theta)^T\|+\|U\cdot\nabla(\nabla\theta)\|+\|\Delta\theta\|\big).\end{aligned} \tag{3.43}$$

Plugging (3.43) into (3.42) we have

$$\begin{aligned}&\lambda^2\|U\cdot\nabla(\nabla\theta)\|^2\\ \le& c_{28}\|U\|\|\nabla U\|\|D^2\theta\|\big(\|\nabla\theta_t\|+\|\nabla U\cdot(\nabla\theta)^T\|+\|U\cdot\nabla(\nabla\theta)\|+\|\theta\|_{H^2}\big).\end{aligned} \tag{3.44}$$

Using Lemma 3.2 and Poincaré inequality we estimate the RHS of (3.44) as follows:

$$\begin{aligned}&c_{28}\|U\|\|\nabla U\|\|D^2\theta\|\big(\|\nabla\theta_t\|+\|\nabla U\cdot(\nabla\theta)^T\|+\|U\cdot\nabla(\nabla\theta)\|+\|\theta\|_{H^2}\big)\\ \le& c_{29}\|\nabla U\|\|\theta\|_{H^2}\big(\|\nabla\theta_t\|+\|\nabla U\cdot(\nabla\theta)^T\|+\|U\cdot\nabla(\nabla\theta)\|\big)+c_{30}\|\nabla U\|^2\|\theta\|_{H^2}^2\\ \le& \frac{\delta}{32c_1}\|\nabla\theta_t\|^2+\frac{\lambda^2}{2}\|U\cdot\nabla(\nabla\theta)\|^2+\frac{c_{31}(\delta)}{2\delta}\|\nabla U\|^2\|\theta\|_{H^2}^2+\frac{1}{2}\|\nabla U\cdot(\nabla\theta)^T\|^2.\end{aligned}$$

Combining the preceding estimate with (3.44) we have

$$\|F_4\|^2 \le \frac{\delta}{16c_1}\|\nabla\theta_t\|^2+\frac{c_{31}(\delta)}{\delta}\|\nabla U\|^2\|\theta\|_{H^2}^2+\|\nabla U\cdot(\nabla\theta)^T\|^2. \tag{3.45}$$

In a similar fashion as deriving (3.41) we have

$$\|\nabla U\cdot(\nabla\theta)^T\|^2 \le \frac{c_{32}}{\varepsilon}\|\nabla U\|^2\|\theta\|_{H^2}^2+\frac{\varepsilon}{48c_1}\|U\|_{H^2}^2,$$

which, together with (3.45), yields

$$\|F_4\|^2 \le \frac{\delta}{16c_1}\|\nabla\theta_t\|^2 + \frac{\varepsilon}{48c_1}\|U\|_{H^2}^2 + \left(\frac{c_{31}(\delta)}{\delta} + \frac{c_{32}}{\varepsilon}\right)\|\nabla U\|^2\|\theta\|_{H^2}^2. \quad (3.46)$$

Finally, using Lemma 2.4 (i) and the condition on $\vec{f}$ we have

$$\|F_5\|^2 = \|\vec{f}\theta\|^2 \le \|\vec{f}\|_{L^4}^2\|\theta\|_{L^4}^2 \le c_4^2 F_1\|\theta\|_{H^1}^2. \quad (3.47)$$

Collecting (3.39)–(3.41) and (3.46)–(3.47) and using (3.38) we complete the proof.

e) Decay of $\|U\|_{H^1}$

With the help of Lemma 3.4 we show the decay of $\|\nabla U\|$ and $\|\theta_t\|$.

Lemma 3.5. *Under the assumptions of Theorem 1.1, there exist positive constants* α_3, β_3 *and* γ_3 *independent of* t *such that for any* $t \ge 0$ *it holds that*

$$\|(\nabla U, \theta_t)(\cdot, t)\|^2 \le \alpha_3 e^{-\beta_3 t}, \quad \text{and} \quad \int_0^t \|(\nabla\theta_t, U_t)(\cdot, \tau)\|^2 d\tau \le \gamma_3.$$

Proof. Taking L^2 inner product of $(3.1)_1$ with U_t we have

$$\frac{\mu}{2}\frac{d}{dt}\|\nabla U\|^2 + \int_\Omega \rho|U_t|^2 d\mathbf{x} = -\int_\Omega \rho(U\cdot\nabla U)U_t d\mathbf{x} + \lambda\int_\Omega \big[\nabla\theta\cdot\nabla U + U\cdot\nabla(\nabla\theta)\big]U_t d\mathbf{x} + \int_\Omega \theta\vec{f}\cdot U_t d\mathbf{x}. \quad (3.48)$$

We estimate the RHS of (3.48) as follows: By Cauchy-Schwarz inequality we have

$$\left| -\int_\Omega \rho(U\cdot\nabla U)U_t d\mathbf{x} + \lambda\int_\Omega \big[\nabla\theta\cdot\nabla U + U\cdot\nabla(\nabla\theta)\big]U_t d\mathbf{x} + \int_\Omega \theta\vec{f}\cdot U_t d\mathbf{x}\right|$$
$$\le \frac{m}{8}\|U_t\|^2 + \frac{2}{m}\big\|\big(\rho U\cdot\nabla U + \lambda\nabla\theta\cdot\nabla U + \lambda U\cdot\nabla(\nabla\theta) + \vec{f}\theta\big)\big\|^2. \quad (3.49)$$

For the second term on the RHS of (3.49), it follows from the proof of Lemma 3.4 that

$$\frac{2}{m}\big\|\big(\rho U\cdot\nabla U + \lambda\nabla\theta\cdot\nabla U + \lambda U\cdot\nabla(\nabla\theta) + \vec{f}\theta\big)\big\|^2$$
$$\le \frac{\lambda}{8}\|\nabla\theta_t\|^2 + \varepsilon_1\|U\|_{H^2}^2 + c_{33}(\varepsilon_1)\big(\|\nabla U\|^2\|\theta\|_{H^2}^2 + \|\nabla U\|^4 + \|\theta\|_{H^1}^2\big),$$

where $\varepsilon_1 > 0$ is a constant to be determined. So we update (3.48) as

$$\frac{\mu}{2}\frac{d}{dt}\|\nabla U\|^2 + \int_\Omega \rho|U_t|^2 d\mathbf{x} \le \frac{m}{8}\|U_t\|^2 + \frac{\lambda}{8}\|\nabla\theta_t\|^2 + \varepsilon_1\|U\|_{H^2}^2 + c_{33}(\varepsilon_1)\big(\|\nabla U\|^2\|\theta\|_{H^2}^2 + \|\nabla U\|^4 + \|\theta\|_{H^1}^2\big). \quad (3.50)$$

Letting $\varepsilon = 1/2$ and $\delta = 1$ in Lemma 3.4 we have

$$\|U\|_{H^2}^2 \le c_{34}\big(\|\nabla U\|^2\|\theta\|_{H^2}^2 + \|\nabla U\|^4 + \|U_t\|^2 + \|\theta\|_{H^1}^2 + \|\nabla\theta_t\|^2\big). \quad (3.51)$$

Notes

Global Journal of Science Frontier Research (F) Volume XII Issue II Version I February 2012

Plugging (3.51) into (3.50) we have

$$\frac{\mu}{2}\frac{d}{dt}\|\nabla U\|^2 + \int_\Omega \rho |U_t|^2 d\mathbf{x} \leq \frac{m}{8}\|U_t\|^2 + \frac{\lambda}{8}\|\nabla\theta_t\|^2 + \varepsilon_1 c_{34}\big(\|U_t\|^2 + \|\nabla\theta_t\|^2\big) + \big(c_{33}(\varepsilon_1) + c_{34}\big)\big(\|\nabla U\|^2\|\theta\|_{H^2}^2 + \|\nabla U\|^4 + \|\theta\|_{H^1}^2\big).$$

Choosing $\varepsilon_1 = \min\{m/(8c_{34}), \lambda/(8c_{34})\}$ and using the fact that $\rho \geq m$ we have

$$\frac{\mu}{2}\frac{d}{dt}\|\nabla U\|^2 + \frac{3m}{4}\|U_t\|^2 \leq \frac{\lambda}{4}\|\nabla\theta_t\|^2 + c_{35}\big(\|\nabla U\|^2\|\theta\|_{H^2}^2 + \|\nabla U\|^4 + \|\theta\|_{H^1}^2\big). \tag{3.52}$$

Notes

Next, by taking the temporal derivative of $(3.1)_2$ we have

$$\theta_{tt} + U_t \cdot \nabla\theta + U \cdot \nabla\theta_t = \lambda\Delta\theta_t. \tag{3.53}$$

Taking L^2 inner product of (3.53) with θ_t we have

$$\frac{1}{2}\frac{d}{dt}\|\theta_t\|^2 + \lambda\|\nabla\theta_t\|^2 = -\int_\Omega (U_t \cdot \nabla\theta)\theta_t d\mathbf{x}. \tag{3.54}$$

Using Cauchy-Schwarz inequality we have

$$\begin{aligned}\left| -\int_\Omega (U_t \cdot \nabla\theta)\theta_t d\mathbf{x}\right| &\leq \frac{m}{4}\|U_t\|^2 + \frac{1}{m}\|(\nabla\theta)\theta_t\|^2 \\ &\leq \frac{m}{4}\|U_t\|^2 + \frac{1}{m}\|\nabla\theta\|_{L^4}^2\|\theta_t\|_{L^4}^2.\end{aligned} \tag{3.55}$$

For the RHS of (3.55), by Lemma 2.4 (iii) and Lemma 3.3 we have

$$\begin{aligned}\frac{1}{m}\|\nabla\theta\|_{L^4}^2\|\theta_t\|_{L^4}^2 &\leq c_{36}(\|\nabla\theta\|\|D^2\theta\| + \|\nabla\theta\|^2)(\|\theta_t\|\|\nabla\theta_t\| + \|\theta_t\|^2) \\ &\leq c_{37}\big(\|D^2\theta\| + \|\nabla\theta\|\big)\|\theta_t\|\|\nabla\theta_t\| + c_{36}\|\theta\|_{H^2}^2\|\theta_t\|^2 \\ &\leq \frac{\lambda}{4}\|\nabla\theta_t\|^2 + c_{38}\|\theta\|_{H^2}^2\|\theta_t\|^2.\end{aligned} \tag{3.56}$$

Combining (3.54)–(3.56) we have

$$\frac{1}{2}\frac{d}{dt}\|\theta_t\|^2 + \frac{3\lambda}{4}\|\nabla\theta_t\|^2 \leq \frac{m}{4}\|U_t\|^2 + +c_{38}\|\theta\|_{H^2}^2\|\theta_t\|^2. \tag{3.57}$$

Coupling (3.52) and (3.57) we have

$$\begin{aligned}&\frac{d}{dt}\big(\mu\|\nabla U\|^2 + \|\theta_t\|^2\big) + m\|U_t\|^2 + \lambda\|\nabla\theta_t\|^2 \\ &\leq c_{39}\big(\|\nabla U\|^2\|\theta\|_{H^2}^2 + \|\nabla U\|^4 + \|\theta\|_{H^1}^2 + \|\theta\|_{H^2}^2\|\theta_t\|^2\big) \\ &\leq c_{40}\big(\|\theta\|_{H^2}^2 + \|\nabla U\|^2\big)\big(\mu\|\nabla U\|^2 + \|\theta_t\|^2\big) + c_{39}\|\theta\|_{H^1}^2.\end{aligned} \tag{3.58}$$

Applying Gronwall inequality to (3.58) and using Lemma 3.2 and Lemma 3.3 we have

$$\mu\|\nabla U\|^2 + \|\theta_t\|^2 \leq c_{41}, \quad \text{and} \quad \int_0^t m\|U_t\|^2 + \lambda\|\nabla\theta_t\|^2 d\tau \leq c_{42}. \tag{3.59}$$

Plugging the first part of (3.59) into (3.58) we have

$$\frac{d}{dt}\big(\mu\|\nabla U\|^2 + \|\theta_t\|^2\big) + m\|U_t\|^2 + \lambda\|\nabla\theta_t\|^2 \leq c_{43}\big(\|\theta\|_{H^2}^2 + \|\nabla U\|^2\big). \tag{3.60}$$

To show the exponential decay of $\|\nabla U\|$ and $\|\theta_t\|$, we consider the estimate (3.30). By absorbing the RHS of (3.60) into the LHS of (3.30) we have

$$\frac{d}{dt}\big(E_2(t)\big) + c_{44}D_2(t) \le 0, \tag{3.61}$$

for some constant $c_{44} > 0$ independent of t, where, by virtue of Poincaré inequality,

$$E_2(t) \cong \|(U,\theta)(\cdot,t)\|_{H^1}^2 + \|\theta_t(\cdot,t)\|^2,$$
$$D_2(t) \cong \|(U,\theta_t)(\cdot,t)\|_{H^1}^2 + \|\theta(\cdot,t)\|_{H^2}^2 + \|U_t(\cdot,t)\|^2.$$

Here $\cong$ denotes the equivalence of quantities. Then the lemma follows immediately from (3.61) and (3.59). This completes the proof.

f) Decay of $\|\theta\|_{H^2}$

Lemma 3.6. *Under the assumptions of Theorem 1.1, there exist constants $\alpha_4, \beta_4, \gamma_4 > 0$ independent of t such that for any $t \ge 0$ it holds that*

$$\|\theta(\cdot,t)\|_{H^2}^2 \le \alpha_4 e^{-\beta_4 t}, \quad \text{and} \quad \int_0^t \|U(\cdot,\tau)\|_{H^2}^2 d\tau \le \gamma_4.$$

Proof. We note that, by Lemma 2.3, Lemma 2.4 and Lemma 3.5 it holds that

$$\begin{aligned}\|\theta\|_{H^2}^2 \le c_3\|\Delta\theta\|^2 &\le c_{45}\big(\|\theta_t\|^2 + \|U\cdot\nabla\theta\|^2\big)\\ &\le c_{46}\big(\|\theta_t\|^2 + \|U\|_{H^1}^2(\|\nabla\theta\|\|\theta\|_{H^2} + \|\nabla\theta\|^2)\big)\\ &\le c_{47}\big(\|\theta_t\|^2 + \|\nabla\theta\|^2\big) + \frac{1}{2}\|\theta\|_{H^2}^2,\end{aligned}$$

which implies that

$$\|\theta\|_{H^2}^2 \le c_{48}\big(\|\theta_t\|^2 + \|\nabla\theta\|^2\big). \tag{3.62}$$

Then the exponential decay of $\|\theta\|_{H^2}^2$ follows from Lemma 3.3 and Lemma 3.5.

Next, by (3.51) and Lemma 3.5 we have

$$\begin{aligned}\|U\|_{H^2}^2 &\le c_{34}\big(\|\nabla U\|^2\|\theta\|_{H^2}^2 + \|\nabla U\|^4 + \|U_t\|^2 + \|\theta\|_{H^1}^2 + \|\nabla\theta_t\|^2\big)\\ &\le c_{49}\big(\|\theta\|_{H^2}^2 + \|\nabla U\|^2 + \|U_t\|^2 + \|\nabla\theta_t\|^2\big),\end{aligned} \tag{3.63}$$

which, together with Lemmas 3.2, 3.3 and 3.5, implies that

$$\int_0^t \|U(\cdot,\tau)\|_{H^2}^2 d\tau \le c_{50}.$$

This completes the proof.

g) Decay of $\|\theta\|_{H^3}$ and $\|U\|_{H^2}$

Lemma 3.7. *Under the assumptions of Theorem 1.1, there exist positive constants α_5, β_5 and γ_5 independent of t such that for any $t \ge 0$ it holds that*

$$\|U(\cdot,t)\|_{H^2}^2 + \|(\nabla\theta_t, U_t)(\cdot,t)\|^2 \le \alpha_5 e^{-\beta_5 t}, \quad \text{and} \quad \int_0^t \|(\nabla U_t, \Delta\theta_t)(\cdot,\tau)\|_{H^2}^2 d\tau \le \gamma_5.$$

Proof. Taking the temporal derivative of $(3.1)_1$ we have

$$\theta_t(U_t + U\cdot\nabla U) + \rho(U_{tt} + U_t\cdot\nabla U + U\cdot\nabla U_t) + \nabla P_t \tag{3.64}$$

$$=\mu\Delta U_t+\lambda\big(\nabla\theta_t\cdot\nabla U+\nabla\theta\cdot\nabla U_t+U_t\cdot\nabla(\nabla\theta)+U\cdot\nabla(\nabla\theta_t)\big)+\vec{f}\theta_t+\vec{f}_t\theta.$$

Taking L^2 inner product of (3.64) with U_t, after integration by parts, we have

$$\frac{1}{2}\frac{d}{dt}\|\sqrt{\rho}U_t\|^2+\mu\|\nabla U_t\|^2+\frac{1}{2}\int_\Omega(\theta_t-U\cdot\nabla\theta)|U_t|^2d\mathbf{x}=\sum_{i=1}^{7}R_i+\lambda\int_\Omega(\nabla\theta\cdot\nabla U_t)\cdot U_td\mathbf{x},$$

where

$$R_1=-\int_\Omega(\theta_tU\cdot\nabla U)\cdot U_td\mathbf{x},\quad R_2=-\int_\Omega(\rho U_t\cdot\nabla U)\cdot U_td\mathbf{x};$$
$$R_3=\lambda\int_\Omega(\nabla\theta_t\cdot\nabla U)\cdot U_td\mathbf{x},\quad R_4=\lambda\int_\Omega(U_t\cdot\nabla(\nabla\theta))\cdot U_td\mathbf{x},$$
$$R_5=-\lambda\int_\Omega\nabla\theta_t\cdot(U\cdot\nabla U_t)d\mathbf{x};$$
$$R_6=\lambda\int_\Omega\theta_t\vec{f}\cdot U_td\mathbf{x},\quad R_7=\lambda\int_\Omega\theta\vec{f}_t\cdot U_td\mathbf{x}.$$

Using the boundary condition we have

$$\lambda\int_\Omega(\nabla\theta\cdot\nabla U_t)\cdot U_td\mathbf{x}=-\frac{\lambda}{2}\int_\Omega\Delta\theta|U_t|^2d\mathbf{x}.$$

Moreover, since $\theta_t=\lambda\Delta\theta-U\cdot\nabla\theta$, we have

$$\frac{1}{2}\frac{d}{dt}\|\sqrt{\rho}U_t\|^2+\mu\|\nabla U_t\|^2=\sum_{i=1}^{9}R_i, \tag{3.65}$$

where

$$R_8=\int_\Omega(U\cdot\nabla\theta)|U_t|^2d\mathbf{x},\quad R_9=-\lambda\int_\Omega\Delta\theta|U_t|^2d\mathbf{x}.$$

We estimate $R_i, i=1,...,9$ as follows: By Lemma 2.4, Lemma 3.5 and Poincaré inequality we have

$$\begin{aligned}|R_1|&\le\|\theta_t\|_{L^4}\|U\|_{L^4}\|\nabla U\|_{L^4}\|U_t\|_{L^4}\\&\le c_{51}\|\theta_t\|_{H^1}\|\nabla U\|_{H^1}\|U_t\|_{H^1}\\&\le c_{52}\|\theta_t\|_{H^1}\|U\|_{H^2}\|\nabla U_t\|\\&\le\varepsilon\|\nabla U_t\|^2+\frac{c_{53}}{\varepsilon}\big(\|\theta_t\|^2+\|\nabla\theta_t\|^2\big)\|U\|^2_{H^2}\\&\le\varepsilon\|\nabla U_t\|^2+\frac{c_{54}}{\varepsilon}\|U\|^2_{H^2}+\frac{c_{53}}{\varepsilon}\|\nabla\theta_t\|^2\|U\|^2_{H^2},\end{aligned}$$

where $\varepsilon>0$ is a constant to be determined. Similarly, we have

$$\begin{aligned}|R_2|&\le\|\rho\|_{L^\infty}\|\nabla U\|\|U_t\|^2_{L^4}\\&\le c_{55}\|U_t\|\|\nabla U_t\|\\&\le\varepsilon\|\nabla U_t\|^2+\frac{c_{56}}{\varepsilon}\|U_t\|^2.\end{aligned}$$

Using Lemma 3.1 and Lemma 3.5 we have

$$|R_3|\le\frac{\lambda}{2}\|\nabla\theta_t\|^2+\frac{\lambda}{2}\|\nabla U\|^2_{L^4}\|U_t\|^2_{L^4}$$

Notes

$$
\begin{aligned}
&\le \frac{\lambda}{2}\|\nabla\theta_t\|^2 + c_{57}\big(\|\nabla U\|\|\nabla^2 U\| + \|\nabla U\|^2\big)\|U_t\|\|\nabla U_t\| \\
&\le \frac{\lambda}{2}\|\nabla\theta_t\|^2 + c_{58}\|\nabla^2 U\|\|U_t\|\|\nabla U_t\| + c_{59}\|U_t\|\|\nabla U_t\| \\
&\le \varepsilon\|\nabla U_t\|^2 + c_{60}(\varepsilon)\big(\|U\|_{H^2}^2\|\sqrt{\rho}U_t\|^2 + \|\nabla\theta_t\|^2 + \|U_t\|^2\big);
\end{aligned}
$$

and

$$
\begin{aligned}
|R_4| &\le \lambda\|\theta\|_{H^2}\|U_t\|_{L^4}^2 \\
&\le c_{61}\|\theta\|_{H^2}\|U_t\|\|\nabla U_t\| \\
&\le \varepsilon\|\nabla U_t\|^2 + \frac{c_{62}}{\varepsilon}\|\theta\|_{H^2}^2\|\sqrt{\rho}U_t\|^2.
\end{aligned}
$$

By Sobolev embedding we have

$$
\begin{aligned}
|R_5| &\le \varepsilon\|\nabla U_t\|^2 + \frac{c_{63}}{\varepsilon}\|U\|_{L^\infty}^2\|\nabla\theta_t\|^2 \\
&\le \varepsilon\|\nabla U_t\|^2 + \frac{c_{64}}{\varepsilon}\|U\|_{H^2}^2\|\nabla\theta_t\|^2.
\end{aligned}
$$

Since $\|\vec{f}_t\|^2_{C([0,t];H^1(\Omega))} + \|\vec{f}_t\|^2_{C([0,t];L^2(\Omega))} \le F_1$, using Poincaré inequality we have

$$
\begin{aligned}
|R_6| &\le \frac{\varepsilon}{c_0}\|U_t\|^2 + \frac{c_{65}}{\varepsilon}\|\vec{f}\|_{L^4}^2\|\theta_t\|_{L^4}^2 \\
&\le \varepsilon\|\nabla U_t\|^2 + \frac{c_{66}}{\varepsilon}\|\theta_t\|_{H^1}^2,
\end{aligned}
$$

and

$$
\begin{aligned}
|R_7| &\le \frac{\varepsilon}{c_0}\|U_t\|^2 + \frac{c_{67}}{\varepsilon}\|\vec{f}_t\|^2\|\theta\|_{L^\infty}^2 \\
&\le \varepsilon\|\nabla U_t\|^2 + \frac{c_{68}}{\varepsilon}\|\theta\|_{H^2}^2.
\end{aligned}
$$

The last two terms are treated as

$$
\begin{aligned}
|R_8| &\le \|U\cdot\nabla\theta\|\|U_t\|_{L^4}^2 \\
&\le c_{69}\|U\|_{L^4}\|\nabla\theta\|_{L^4}\|U_t\|\|\nabla U_t\| \\
&\le c_{70}\|\theta\|_{H^2}\|U_t\|\|\nabla U_t\| \\
&\le \varepsilon\|\nabla U_t\|^2 + \frac{c_{71}}{\varepsilon}\|\theta\|_{H^2}^2\|\sqrt{\rho}U_t\|^2;
\end{aligned}
$$

and

$$
\begin{aligned}
|R_9| &\le \lambda\|\Delta\theta\|\|U_t\|_{L^4}^2 \\
&\le c_{72}\|\theta\|_{H^2}\|U_t\|\|\nabla U_t\| \\
&\le \varepsilon\|\nabla U_t\|^2 + \frac{c_{73}}{\varepsilon}\|\theta\|_{H^2}^2\|\sqrt{\rho}U_t\|^2.
\end{aligned}
$$

Plugging above estimates into (3.65) we have

$$
\frac{1}{2}\frac{d}{dt}\|\sqrt{\rho}U_t\|^2 + \mu\|\nabla U_t\|^2 \le 9\varepsilon\|\nabla U_t\|^2 + K(t)(\|\sqrt{\rho}U_t\|^2 + \|\nabla\theta_t\|^2) + Z(t), \qquad (3.66)
$$

where

$$
K(t) = c_{74}(\varepsilon)\big(\|U\|_{H^2}^2 + \|\theta\|_{H^2}^2\big),
$$

Notes

$$Z(t) = c_{75}(\varepsilon)\big(\|U_t\|^2 + \|U\|_{H^2}^2 + \|\theta_t\|_{H^1}^2 + \|\theta\|_{H^2}^2\big).$$

Next, taking L^2 inner product of (3.53) with $\Delta\theta_t$ we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla\theta_t\|^2 + \lambda\|\Delta\theta_t\|^2 &= \int_\Omega (U_t\cdot\nabla\theta + U\cdot\nabla\theta_t)\Delta\theta_t d\mathbf{x}\\ &\le \frac{\lambda}{2}\|\Delta\theta_t\|^2 + \lambda\big(\|U_t\cdot\nabla\theta\|^2 + \|U\cdot\nabla\theta_t\|^2\big).\end{aligned} \tag{3.67}$$

Notes

The second term on the RHS of (3.67) is estimated as

$$\begin{aligned}&\lambda\big(\|U_t\cdot\nabla\theta\|^2 + \|U\cdot\nabla\theta_t\|^2\big)\\ &\le c_{76}\|U_t\|_{L^4}^2\big(\|\nabla\theta\|\|D^2\theta\| + \|\nabla\theta\|^2\big) + \lambda\|U\|_{L^\infty}^2\|\nabla\theta_t\|^2\\ &\le c_{77}\|U_t\|\|\nabla U_t\|\|\theta\|_{H^2} + c_{78}\|U\|_{H^2}^2\|\nabla\theta_t\|^2\\ &\le \varepsilon\|\nabla U_t\|^2 + \frac{c_{79}}{\varepsilon}\|\theta\|_{H^2}^2\|\sqrt{\rho}U_t\|^2 + c_{78}\|U\|_{H^2}^2\|\nabla\theta_t\|^2.\end{aligned}$$

It follows that

$$\frac{1}{2}\frac{d}{dt}\|\nabla\theta_t\|^2 + \frac{\lambda}{2}\|\Delta\theta_t\|^2 \le \varepsilon\|\nabla U_t\|^2 + \frac{c_{79}}{\varepsilon}\|\theta\|_{H^2}^2\|\sqrt{\rho}U_t\|^2 + c_{78}\|U\|_{H^2}^2\|\nabla\theta_t\|^2. \tag{3.68}$$

Combining (3.66) and (3.68) we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\big(\|\sqrt{\rho}U_t\|^2 + \|\nabla\theta_t\|^2\big) + \mu\|\nabla U_t\|^2 + \frac{\lambda}{2}\|\Delta\theta_t\|^2\\ &\le 10\varepsilon\|\nabla U_t\|^2 + \tilde{K}(t)\big(\|\sqrt{\rho}U_t\|^2 + \|\nabla\theta_t\|^2\big) + \tilde{Z}(t),\end{aligned} \tag{3.69}$$

where $\tilde{K}(t)$ and $\tilde{Z}(t)$ are equivalent to $K(t)$ and $Z(t)$ respectively. Choosing $\varepsilon = \mu/20$ in (3.69) we have

$$\frac{d}{dt}\big(\|\sqrt{\rho}U_t\|^2+\|\nabla\theta_t\|^2\big)+\mu\|\nabla U_t\|^2+\lambda\|\Delta\theta_t\|^2 \le 2\tilde{K}(t)(\|\sqrt{\rho}U_t\|^2+\|\nabla\theta_t\|^2)+2\tilde{Z}(t). \tag{3.70}$$

By virtue of Lemmas 3.5–3.6 we know that $\tilde{K}(t), \tilde{Z}(t)$ are uniformly integrable in time for any $t \ge 0$. Applying Gronwall inequality to (3.70) we have

$$\|(\sqrt{\rho}U_t, \nabla\theta_t)(\cdot,t)\|^2 \le c_{79}, \quad \text{and} \quad \int_0^t \|(\nabla U_t, \Delta\theta_t)(\cdot,\tau)\|^2 d\tau \le c_{80}, \quad \forall\ t \ge 0. \tag{3.71}$$

Plugging the first part of (3.71) into (3.70) we have

$$\frac{d}{dt}\big(\|\sqrt{\rho}U_t\|^2 + \|\nabla\theta_t\|^2\big) + \mu\|\nabla U_t\|^2 + \lambda\|\Delta\theta_t\|^2 \le c_{81}Y(t), \tag{3.72}$$

where

$$Y(t) = \|U_t\|^2 + \|U\|_{H^2}^2 + \|\theta_t\|_{H^1}^2 + \|\theta\|_{H^2}^2.$$

By virtue of (3.63), Poincaré inequality and Lemma 2.3 we have

$$Y(t) \le c_{82}\big(\|U_t\|^2 + \|\nabla U\|^2 + \|\nabla\theta_t\|^2 + \|\Delta\theta\|^2\big). \tag{3.73}$$

Plugging (3.73) into (3.72) we have

$$\frac{d}{dt}\big(\|\sqrt{\rho}U_t\|^2 + \|\nabla\theta_t\|^2\big) + \mu\|\nabla U_t\|^2 + \lambda\|\Delta\theta_t\|^2 \le c_{83}\|(U_t, \nabla U, \nabla\theta_t, \Delta\theta)\|^2. \tag{3.74}$$

Global Journal of Science Frontier Research (F) Volume XII Issue II Version I February 2012

where

$$E_3(t) \cong \|(\theta, \theta_t, U)\|_{H^1}^2 + \|U_t\|^2,$$
$$D_3(t) \cong \|(\theta, \theta_t)\|_{H^2}^2 + \|(U, U_t)\|_{H^1}^2.$$

Then the lemma follows directly from (3.63), (3.71), (3.75) and Lemma 3.6. This completes the proof.

As a consequence of Lemma 3.7 we have

Lemma 3.8. *Under the assumptions of Theorem 1.1, there exist positive constants* α_6 *and* β_6 *independent of* t *such that for any* $t \geq 0$ *it holds that*

$$\|\theta(\cdot, t)\|_{H^3}^2 \leq \alpha_6 e^{-\beta_6 t}.$$

Proof. By virtue of Lemma 2.3 we have

$$\begin{aligned}\|\theta\|_{H^3}^2 \leq c_3\|\Delta\theta\|_{H^1}^2 &\leq c_{85}\big(\|\Delta\theta\|^2 + \|\nabla\theta_t\|^2 + \|\nabla(U\cdot\nabla\theta)\|^2\big)\\ &\leq c_{86}\big(\|\Delta\theta\|^2 + \|\nabla\theta_t\|^2 + \|U\|_{H^2}^2\|\theta\|_{H^2}^2\big).\end{aligned}$$

Then the lemma follows from Lemma 3.6 and Lemma 3.7. This completes the proof.

h) Decay of $\|U\|_{H^3}$

Lemma 3.9. *Under the assumptions of Theorem 1.1, there exist positive constants* α_7, β_7 *and* γ_6 *independent of* t *such that for any* $t \geq 0$ *it holds that*

$$\|U(\cdot, t)\|_{H^3}^2 \leq \alpha_7 e^{-\beta_7 t}, \quad \text{and} \int_0^t \big(\|\theta_t(\cdot, \tau)\|_{H^2}^2 + \|U_{tt}(\cdot, \tau)\|^2\big)d\tau \leq \gamma_6.$$

Proof. Taking L^2 inner product of (3.64) with U_{tt} we have

$$\begin{aligned}&\frac{\mu}{2}\frac{d}{dt}\|\nabla U_t\|^2 + \|\sqrt{\rho}U_{tt}\|^2\\ &= \int_\Omega \big[-\rho_t U_t - \rho_t U\cdot\nabla U - \rho U_t\cdot\nabla U - \rho U\cdot\nabla U_t\\ &+\lambda\big(\nabla\rho_t\cdot\nabla U + \nabla\rho\cdot\nabla U_t + U_t\cdot\nabla(\nabla\rho) + U\cdot\nabla(\nabla\rho_t)\big) + \vec{f}\rho_t + \vec{f}_t\rho\big]\cdot U_{tt}d\mathbf{x}.\end{aligned} \quad (3.76)$$

Using previously established estimates and Lemma 2.4, we can show that (since there is no essential difficulties, we omit the details)

$$\frac{\mu}{2}\frac{d}{dt}\|\nabla U_t\|^2 + \frac{1}{2}\|\sqrt{\rho}U_{tt}\|^2 \leq c_{87}\big(\|\nabla U_t\|^2 + \|\theta_t\|_{H^2}^2 + \|\theta\|_{H^2}^2\big). \quad (3.77)$$

By absorbing the RHS of (3.77) into the LHS of (3.75) we have

$$\frac{d}{dt}E_4(t) + c_{88}D_4(t) \leq 0, \quad \forall\ t \geq 0, \quad (3.78)$$

where

$$E_4(t) \cong \|(U, U_t, \theta, \theta_t)\|_{H^1}^2,$$
$$D_4(t) \cong \|(\theta, \theta_t)\|_{H^2}^2 + \|(U, U_t)\|_{H^1}^2 + \|U_{tt}\|^2.$$

So that, for any $t \geq 0$ it holds that

$$\|U_t(\cdot, t)\|_{H^1}^2 \leq c_{89}e^{-c_{90}t}, \quad \text{and} \int_0^t \big(\|\theta_t(\cdot, \tau)\|_{H^2}^2 + \|U_{tt}(\cdot, \tau)\|^2\big)d\tau \leq c_{91}. \quad (3.79)$$

Notes

With the help of previous estimates and Lemma 2.1, by direct calculations, we have

$$\|U\|_{H^3}^2 \le c_{92}\big(\|U\|_{H^2}^2 + \|\theta\|_{H^3}^2 + \|U_t\|_{H^1}^2\big).$$

Then the lemma follows from Lemma 3.7, Lemma 3.8 and (3.79). This completes the proof.

i) Uniform estimate of $\|(\theta, U)\|_{H^4}$

Notes

We now prove the uniform estimates of $\|(\theta, U)\|_{H^4}$ in order to complete the proof of Theorem 1.1.

Lemma 3.10. *Under the assumptions of Theorem 1.1, there exists a positive constant* γ_7 *independent of* t *such that for any* $t \ge 0$ *it holds that*

$$\int_0^t \|(U,\theta)(\cdot,t)\|_{H^4}^2 d\tau \le \gamma_7, \quad \forall\ t \ge 0.$$

Proof. By Lemma 2.3, Lemma 2.1 and Lemma 3.9, it is straightforward to show that

$$\begin{aligned} \|\theta\|_{H^4}^2 &\le c_{93}\big(\|\theta_t\|_{H^2}^2 + \|\theta\|_{H^3}^2\big), \\ \|U\|_{H^4}^2 &\le c_{94}\big(\|U_t\|_{H^2}^2 + \|\theta\|_{H^4}^2\big). \end{aligned} \tag{3.80}$$

Since $U_t|_{\partial\Omega} = 0$, by Lemma 2.1 and (3.64) we have

$$\|U_t\|_{H^2}^2 \le c_{95}\big(\|U_{tt}\|^2 + \|\rho_t\|_{H^2}^2 + \|U\|_{H^3}^2\|\rho\|_{H^3}^2\big). \tag{3.81}$$

Then the lemma follows from Lemma 3.9, (3.80) and (3.81). This completes the proof.

Lemmas 3.8–3.10 conclude our main result, Theorem 1.1.

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Notes